数理解析研究所講究録
657巻 1988年 45-60

On Legendre's theorem related to Diophantine approximation

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§1 Introduction

It is well known that the sequeuce of convergents p_n / q_n of α, which is induced by a continued faction expansion, provides "good" approximation of α.

More exactly we have the following theorem.

<u>Theorem</u> (Legendre). *Let* α *be a positive irrational number. if a simple fraction* p/q, q>0, *satisfies the inequality*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{2} \frac{1}{q^2}$$

then there exists n *such that*

$$(p_n, q_n) = (p, q)$$

and the constant 1/2 *is best possible.*

We call the best possible constant 1/2 Legendre constant associated with a simple continued fraction expension or Legendre constant related to the principle convergents.

By the way we know other continued fraction expensions, for example nearest integer continued faction expansion, α-continued fraction expansion. and so forth. And each expansion provides a "good" approximation of α.

It is the main purpose of this paper to discuss how we are able to dicide the Legendre constant for each continued faction expansion.

In this paper, we give the proof in the case of the simple continued fraction expansion, but it is easy to see that the mithod of the proof is universal and it is able to be extended to other expansions. The central idea consists in constructing a kind of decomposition of integer lattice, which will be called cone decomposition associated with a simple continued fraction expansion, and main tool is the natural extension of the simple continued fraction expansion transformation, which is basic concept in ergodic theory.

In the final section, we give Legendre constants for several kinds of continued fraction expansions.

§2 On the simple continued fraction expansion

Let $X=[0,1]$ and an integer valued function $a(\alpha)$ on X be $a(\alpha)=[1/\alpha]$, and define a map T on X and digits $a_n (n\geqq 1)$ by

$$T\alpha = \frac{1}{\alpha} - a(\alpha) \tag{2.1}$$

and

$$a_n = a_n(\alpha) = a(\, T^{n-1}\alpha\,) , \qquad n \geqq 1 ,$$

then, by using the algorithm (X,T), we have a simple continued fraction expansion of $\alpha\in X$ inductively:

$$\alpha = \frac{1}{\alpha_1 + T\alpha} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + T^n\alpha}}} \tag{2.2}$$

Put

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad (n \geqq 1)$$

and

$$\begin{pmatrix} q_0 & q_{-1} \\ p_0 & p_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and define a linear transformetion ϕ_k from (x_k, y_k) plane to (x_{k-1}, y_{k-1}) plane by

$$\phi_k : \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} , \qquad (2.3)$$

then we see inductively the following fundamental properties.

<u>Fundamental property 1.1.</u> Let us denote $\alpha_n = T^n \alpha$, then for each irrational $\alpha \in [0,1)$,

1) $$\alpha = \frac{p_n + p_{n-1}\alpha_n}{q_n + q_{n-1}\alpha_n}$$

2) $$\alpha\, \alpha_1 \cdots \alpha_{n-1} = \frac{1}{q_n + q_{n-1}\alpha_n}$$

and

3) $$\alpha \cdot x_0 - y_0 = (-1)^n \alpha\, \alpha_1 \alpha_2 \cdots \alpha_{n-1} (\alpha_n x_n - y_n) .$$

Define a map $\phi_{n+1,k}$ from (x_{n+1}, y_{n+1}) plane to (x_n, y_n) plane by

$$\phi_{n+1,k} : \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \qquad (2.4)$$

and define a cone L_0 and a lattice cone $\mathcal{L}_0$ in R^2 by

$$L_0=\{x\binom{1}{0}+y\binom{1}{1}:x,y\geq 0\}$$

and

$$\mathcal{L}_0=\{\ell\binom{1}{0}+m\binom{1}{1}:\ell,m\in N\} \quad , \tag{2.5}$$

and also define a cone $L(n,k)$ and a lattice cone $\mathcal{L}(n,k)$ on (x_0,y_0)-plane by

$$L(n,k)=\phi_1\circ\phi_2\circ\cdots\circ\phi_n\circ\phi_{n+1,k}(L_0)$$

and

$$\mathcal{L}(n,k)=\phi_1\circ\phi_2\circ\cdots\circ\phi_n\circ\phi_{n+1,k}(\mathcal{L}_0) \quad ,$$

then we have the following formula :

$$\mathcal{L}(n,k)=\{\ell\{k\begin{pmatrix}q_n\\p_n\end{pmatrix}+\begin{pmatrix}q_{n-1}\\p_{n-1}\end{pmatrix}\}+m\{(k+1)\begin{pmatrix}q_n\\p_n\end{pmatrix}+\begin{pmatrix}q_{n-1}\\p_{n-1}\end{pmatrix}\}:\ell,m\in N\}$$

(see figure(1)) .

Using the above notation, we get the geometrical lemma.

<u>Lemma 2.1.</u> let ℓ_0 be a line $\ell_0 : \alpha\cdot x_0-y_0=0$

then we have

1) $L(n,k)\cap\ell_0=\phi$, if $k\neq a_{n+1}$

2) $L(n,a_{n+1})\supset\ell_0$

3) $L(n,a_{n+1})=\sum_{k=1}^{\infty}L(n+1,k)$

and

4) $\bigcap_{n=1}^{\infty}L(n,a_{n+1})=\ell_0$.

<u>Definition 2.1</u>. for each irrational $\alpha\in(0,1)$,

$\mathcal{L}_0$ has a following decomposition $\mathcal{L}_0(\alpha)$:

$$\mathcal{L}_0(\alpha)=\sum_{n=0}^{\infty}\sum_{\substack{k=1\\k\neq a_{n+1}}}^{\infty}\mathcal{L}(n,k) \tag{2.6}$$

and we call $\mathcal{L}_0(\alpha)$ a cone decomposition associated with the simple continued fraction expansion of α.

Our goal in this section is to prove the Legendre constant is equal to 1/2. For this purpose, it is sufficient to see

$$\inf_{\alpha\in X} \inf_{\substack{(q,p):\\ (q,p)\neq(q_n,p_n)\\ (q,p)=1}} q\left| q\alpha - p\right| = \frac{1}{2},$$

and, by a cone decomposition (2.6), it is equivalent to

$$\inf_{\alpha\in X} \inf_{n} \inf_{k:k\neq a_{n+1}} \inf_{\substack{(q,p)\in\mathcal{L}(n,k):\\ (q,p)=1\\ (q,p)\neq(q_{n+1},p_{n+1})}} q\left|q\alpha-p\right| = \frac{1}{2} .$$

The following lemma is easily obtained from geometric consideration.

<u>Lemma 2.2.</u> for each irrational $\alpha\in(0,1)$,

1) $$\min_{(p,q)\in\mathcal{L}(n,k)} q|q\alpha-p| \geqq x_{n,k}\left|\alpha\cdot x_{n,k}-y_{n,k}\right|$$
$$\text{or } x_{n,k+1}\left|\alpha\cdot x_{n,k+1}-y_{n,k+1}\right|$$

where $(x_{n,\ell},y_{n,\ell})$ is given by

$$\begin{pmatrix} x_{n,\ell}\\ y_{n,\ell}\end{pmatrix} = \ell\begin{pmatrix} q_n\\ p_n\end{pmatrix}+\begin{pmatrix} q_{n-1}\\ p_{n-1}\end{pmatrix}$$

and moreover

2) $$\min_{k:k\neq a_{n+1}} \min_{(p,q)\in\mathcal{L}(n,k)} q|\, q\alpha - p\,|$$
$$= \min_{\ell=1,a_{n+1}-1,a_{n+1}+1}\left|\alpha\cdot x_{n,\ell}-y_{\ell,n}\right| .$$

By the way, from the fundamental properties (1),(2) and (3) we have the following

<u>Lemma 2.3.</u> for each irrational α,

$$x_{n,\ell}\left|\alpha\cdot x_{n,\ell}-y_{n,\ell}\right| = \frac{\left(\ell+\frac{q_{n-1}}{q_n}\right)\left|1-\alpha_n\cdot\ell\right|}{1+\frac{q_{n-1}}{q_n}\cdot\alpha_n} .$$

Remark 2.1. we know from the definition of a continued fraction expansion that

1) if $a_{n+1} = 1$ then $(x_{n,1}, y_{n,1}) = (q_{n+1}, p_{n+1})$

2) if $a_{n+1} = 2$ then $(x_{n,1}, y_{n,1}) = (x_{n,a_{n+1}-1}, y_{n,a_{n+1}-1})$

3) $$\operatorname{sign}(1-\alpha_n \cdot \ell) = \begin{cases} 1 & \text{if} \quad 1 \leqq \ell \leqq a_{n+1} \\ -1 & \text{if} \quad \ell \geqq a_{n+1}+1 \end{cases} .$$

Using Lemma 2.3 and the Remark 2.1, we define the constants as follows:

$$c_1(\alpha) := \min_{n: a_{n+1} \neq 1} \frac{\left(1+\frac{q_{n+1}}{q_n}\right)(1-\alpha_n)}{1+\frac{q_{n-1}}{q_n}\cdot\alpha} \tag{2.7}$$

$$c_2(\alpha) := \min_{n: a_{n+2} > 2} \frac{\left((a_{n+1}-1)+\frac{q_{n+1}}{q_n}\right)(1-(a_{n+1}-1)\alpha_n)}{1+\frac{q_{n-1}}{q_n}\cdot\alpha_n}$$

$$c_3(\alpha) := \min_{n} \frac{\left((a_{n+1}+1)+\frac{q_{n+1}}{q_n}\right)(\alpha(a_{n+1}+1)-1)}{1+\frac{q_{n-1}}{q_n}\cdot\alpha_n}$$

and

$$c_0(\alpha) := \min_{i=1,2,3.} c_i(\alpha) ,$$

and

$$c_0 = \min_{\alpha\in[0,1)} c_0(\alpha) .$$

Then we have from Lemma 2.2 and 2.3

Proposition 2.1. If there exists a simple fraction p/q such that

$$q|q\alpha-p|<c_0$$

then there exists n satisfying $(q_n, p_n)=(q,p)$ and the constant c_0 is best possible, that is, constant c_0 is the Legendre constant associated with the simple continued fraction.

To calculate the value of the constant c_0 we use the following transformation $(\overline{X},\overline{T})$, which is called a natural extension of a simple continued fraction algorithm (X,T).

Let $\overline{X}=[0,1)\times[0,1)$ and define the map $\overline{T}$ on $\overline{X}$

by $\quad \overline{T}(\alpha,\beta)=\left(T\alpha,\ \frac{1}{\alpha_1+\beta}\right)$, then we obtain the fundamental Lemma. (The proof is found in [3].)

<u>Fundamental Lemma 2.5.</u> For any irrational $\alpha\in[0,1)$,

$$\overline{T}^n(\alpha,0)=\left(\alpha_n,\ \frac{q_{n-1}}{q_n}\right) \qquad (2.8)$$

Therefore the value of $f\left(\alpha_n,\ \frac{q_{n-1}}{q_n}\right)$ is equal to $f\left(\overline{T}^n(\alpha,0)\right)$ for any function f on $\overline{X}$.

<u>Theorem</u> (Legendre)

The value of the constant c_0 *is equal to* 1/2.

Proof. It is not difficult to see that

$$c_1(\alpha) < c_2(\alpha) < c_3(\alpha) .$$

Therefore we try to calculate the constant $c_1=\min c_1(\alpha)$.

From the formula (2.7) and fundamental lemma 2.5, we see

$$c_1(\alpha) = \min_{n:a_{n+1}\neq 1} f(\overline{T}^n(\alpha,0)) \geqq \min_{\substack{(x,y)\in\overline{X}\\ :x<1/2}} f(x,y)$$

where $f(x,y)=\frac{(1+y)(1-x)}{1+xy}$.

Now it is easy to see that

$$c_1\geqq \min_{\substack{(x,y)\in\overline{X}\\ :x<1/2}} f(x,y)=\min\{f(0,0),f(0,1),f\left(\tfrac{1}{2},0\right),f\left(\tfrac{1}{2},1\right)\} = f\left(\tfrac{1}{2},0\right) = \tfrac{1}{2} .$$

The opposite inequality, that is, the constant $\frac{1}{2}$ is best possible, is concluded from the property the closure of the orbits $\{\overline{T}^n(\alpha,0):n\geqq1\}$ is equal to $\overline{X}$ for almost all α.

§3 results

In section 2, we had a method to find the Legendre constant. This method is summarized as follows:

1-st step : to find the algorithm (transformation) (X,T) related to the given diophantine approximation,

2-nd step : to decide the cone decomposition (2.6) associated with the algorithm,

3-rd step : to obtain the formula(2.7) in order to calculate the constant c_0,

4-th step : to construct the natural extension $(\overline{X},\overline{T})$ of the algorithm (X,T), and to decide the statement of fundamental lemma(2.8)

5-th step : to calculate the constant by integrating the results of 3-rd and 4-th steps.

In this section, we will see on several examples that the method is "universal" for "any" one dimentional diophantine approximation.

Example 1. <u>an approximation of x from below</u>.

As an approximation of α from below: $\alpha - \frac{p}{q} > 0$ we know the following algorithm.(see[2])

<u>Algorithm</u>: put

$$X=[0,1]$$

$$b(\alpha)=\lceil \frac{1}{\alpha} \rceil \quad \text{(smallest integer greater than } \frac{1}{\alpha}\text{)}$$

$$T(\alpha)=b(\alpha)-\frac{1}{\alpha}$$

and

$$b_i(\alpha)=b(T^{i-1}\alpha),$$

then the algorithm (X,T,b) induces an approximation $\frac{p'_n}{q'_n}$ of α

where (q'_n, p'_n) is defined by

$$\begin{pmatrix} q'_n & -q'_{n-1} \\ p'_n & -p'_{n-1} \end{pmatrix} := \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix} \qquad (n \geq 1) \quad .$$

Cone decomposition : put

$$\phi_n : \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\phi_{n+1,k} : \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \qquad (k \geq 2)$$

and

$$\mathscr{L}_B(n,k) := \phi_1 \phi_2 \cdots \phi_n \phi_{n+1,k}(\mathscr{L}_0)$$

where

$$\mathscr{L}_0 = \{ l \begin{pmatrix} 1 \\ 0 \end{pmatrix} + m \begin{pmatrix} 1 \\ 1 \end{pmatrix} : l, m \in N \} \quad ,$$

then we have a cone decomposition

$$\mathscr{L}_B(\alpha) = \sum_{n=0}^{\infty} \sum_{k > b_{n+1}}^{\infty} \mathscr{L}_B(n,k)$$

of the domain $\{(x,y) : \alpha x - y > 0, x > 0\}$.

Formula of Legendre constant :

$$\min_{\substack{(q,p): \\ (q,p)=1 \\ q\alpha - p > 0 \\ (q,p) \neq (q'_n, p'_n)}} q(q\alpha - p) \geq \min_n \frac{\left(1 + \frac{1}{b_{n+1}} + \frac{1}{b_{n+1}} \cdot \frac{q'_{n-1}}{q'_n} \right)}{\left(1 + \frac{q'_{n-1}}{q'_n} \cdot \alpha_n \right)} \qquad (3.1)$$

Natural extension: put $\overline{X} = [0,1] \times [0,1]$

and define a map $\overline{T}$ on $\overline{X}$ by $\overline{T}(\alpha, \beta) = \left(T\alpha, \frac{1}{b_1 - \beta} \right)$, then $(\overline{X}, \overline{T})$ is

a natural extension of (X,T) and the following formula holds:

$$\overline{T}^n (\alpha, 0) = \left(T^n \alpha, \frac{q'_{n-1}}{q'_n} \right) .$$

Result: using formula (3.1) and natural extension, the value of Legendre constant c_B on the algorithm is calculated as $\min_{(x,y) \in \overline{X}} \frac{1 + \frac{1}{2} + \frac{1}{2} y}{1 + xy}$ and its vale is equal to 1, that is, we have the following result:

if $0<q(q\alpha-p)<1$ $(q,p)=1$ and $q>0$

then there exists n such that

$(q,p)=(q_n,p_n)$.

Example 2. Nearest integer and α-continued fraction expansion.

The nearest integer continued fraction expansion and singular continued fraction expansion are the special cases of α-continued fraction expansions defined as follow. (see[4])

Algorithm: for $\frac{1}{2}\leqq\alpha\leqq 1$, put $X_\alpha:=[\alpha-1,\alpha]$,

$$a(x):=\left[\ \left|\frac{1}{x}\right|+1-\alpha\ \right]$$
$$\varepsilon(x):=\begin{cases}1 & \text{for } x\geqq 0\\ -1 & \text{for } x<0\end{cases}$$
$$T_\alpha x:=\left|\frac{1}{x}\right|-a(x)$$

and

$$a_n(x):=a(T_\alpha^{n-1}x)$$
$$\varepsilon_n(x):=\varepsilon(T_\alpha^{n-1}x)\quad ,$$

then the algorithm $(X_\alpha,T_\alpha,a,\varepsilon)$ $(\frac{1}{2}\leqq\alpha\leqq 1)$ induces an approximation $\frac{p_n(\alpha)}{q_n(\alpha)}$ of x where $(q_n(\alpha),p_n(\alpha))$ is defined by

$$\begin{pmatrix}q_n(\alpha) & q_{n-1}(\alpha)\\ p_n(\alpha) & p_{n-1}(\alpha)\end{pmatrix}=\begin{pmatrix}a_1 & 1\\ \varepsilon_1 & 0\end{pmatrix}\cdots\begin{pmatrix}a_n & 1\\ \varepsilon_n & 0\end{pmatrix}.$$

Cone decomposition : put

$$\phi_{n,\alpha}:\quad \begin{pmatrix}x_{n-1}\\ y_{n-1}\end{pmatrix}=\begin{pmatrix}a_n & 1\\ \varepsilon_n & 0\end{pmatrix}\begin{pmatrix}x_n\\ y_n\end{pmatrix}$$

$$\phi_{n+1,k}:\quad \begin{pmatrix}x_n\\ y_n\end{pmatrix}=\begin{pmatrix}k & 1\\ \varepsilon_{n+1} & 0\end{pmatrix}\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}$$

and

$$\mathcal{L}_\alpha(n,k):=\phi_1,\phi_2,\cdots,\phi_n\phi_{n+1,k}(\mathcal{L}_{0,\varepsilon_{n+1}})$$

where $\mathcal{L}_{0,\varepsilon}$ is given by $\{l\binom{1}{0}+m\binom{1}{\varepsilon}\ \big|\ l,m\in\mathbb{N}\}$ for $\varepsilon=\pm1$,

then we have a cone decomposition

$$\mathcal{L}_\alpha(x)=\sum_{n=0}^{\infty}\sum_{\substack{k=1\\k\neq a_{n+1}}}^{\infty}\mathcal{L}_\alpha(n,k)$$

of the domain $\{(x,y):-x<y<x,x>0\}$

Formula of Legendre constant :

$$\min_{\substack{(q,p):(q,p)=1\\q>0\\(q,p)\neq(q_n(\alpha),p_n(\alpha))}} q|qx-p| \geqq \min_{\substack{n:\\a_{n+1}\neq 1}} \frac{\left(1+\varepsilon_{n+1}\cdot\frac{q_{n-1(\alpha)}}{q_{n(\alpha)}}\right)\left(1-\varepsilon_{n+1}\alpha_n\right)}{1+\frac{q_{n-1(\alpha)}}{q_{n(\alpha)}}\cdot\alpha_n}$$

Natural extension: put

$$\overline{X}_\alpha:=[\alpha-1,\tfrac{1-2\alpha}{\alpha}]\times[0,g^2]\cup\left(\tfrac{1-2\alpha}{\alpha},\ \tfrac{2\alpha-1}{1-\alpha}\right)\times[0,\tfrac{1}{2}]$$
$$\cup[\tfrac{2\alpha-1}{1-\alpha},\alpha]\times[0,g] \qquad (\text{in the case of } \tfrac{1}{2}\leqq\alpha<g)$$

and
$$\overline{X}_\alpha:=[\alpha-1,\ \tfrac{1-\alpha}{\alpha}]\times[0,\tfrac{1}{2}]\cup[\tfrac{1-\alpha}{\alpha},\alpha]\times[0,1]$$
$$(\text{in the case of } g<\alpha<1)$$

where g is given by $g=\frac{\sqrt{5}-1}{2}$,

and define a map $\overline{T}_\alpha$ on $\overline{X}_\alpha$ by

$$\overline{T}_\alpha(x,y)=\left(T_\alpha x,\frac{1}{a_1+y}\right),$$

then $(\overline{X}_\alpha,\overline{T}_\alpha)$ is a natural extension of (X_α,T_α) and the following formula holds:

$$\overline{T}_\alpha{}^n(\alpha,0)=\left(T_\alpha^n x,\frac{q_{n-1}}{q_n}\right) .$$

Result:

The value of Legendre constant $c(\alpha)$ associated with α-continued farction expansion is culculated as:

$$\min_{(x,y)\in\overline{X}_\alpha} f_\pm(x,y) \quad \text{if } \tfrac{1}{2}\leqq a\leqq g$$

and
$$\min_{(x,y)\in\overline{X}_\alpha} f_\pm(x,y) \quad \text{if } g<\alpha<1$$
$$\text{and} \quad x<\frac{1}{\alpha+1}$$

where $f_\pm(x,y)=\frac{(1\pm y)(1\mp x)}{1+xy}$,

and $c(\alpha)$ is given by :

$$c(\alpha)=\begin{cases} \min(1-\alpha,\frac{\alpha}{1+g\alpha}) & \text{if } 1/2\leq \alpha \leq g \\ \frac{\alpha}{\alpha+1} & \text{if } g\leq \alpha \leq 1 \end{cases}.$$

In particular the both Legendre constants associated with Nearest integer continued fraction expanion and singular continued fraction expansion, which correspond respectively to the expansion of the value $\alpha=\frac{1}{2}$ and $\alpha=g$, are the same and equal to g^2.

Example 3 <u>Mediant convergent transformation.</u>

<u>Algorithm</u>: put $X:=[0,1]$,

$$T_\alpha x =\begin{cases} \frac{x}{1-x} & \text{if } x\in[0,\frac{1}{2}] \\ \frac{1-x}{x} & \text{if } x\in[\frac{1}{2},1] \end{cases},$$

$$\varepsilon(x):=\begin{cases} 0 & \text{if } x\in[0,\frac{1}{2}] \\ 1 & \text{if } x\in[\frac{1}{2},1] \end{cases},$$

and $\quad \varepsilon_n(x):=\varepsilon(T^{n-1}x)$,

then the algorithm (X,T,ε) induces the approximation $\frac{v_n}{w_n}$, which is a mediant or a principle convergent of x, as follows (see[1]):

let matrixes A_0 and A_1 be

$$A_0=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \text{ and } A_1=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

and put

$$\begin{pmatrix}r_n & s_n\\ t_n & u_n\end{pmatrix}:=A_{\varepsilon_1}A_{\varepsilon_2}\cdots A_{\varepsilon_n},$$

then (v_n, w_n) is given by

$$w_n=t_n+u_n\ ,\ v_n=r_n+s_n\ .$$

<u>Cone decomposition</u>: put

$$\phi_n:\ \begin{pmatrix}x_{n-1}\\ y_{n-1}\end{pmatrix}= A_{\varepsilon_n}\begin{pmatrix}x_n\\ y_n\end{pmatrix}$$

and

$$\mathcal{L}_0=\{l\begin{pmatrix}1\\0\end{pmatrix}+m\begin{pmatrix}1\\1\end{pmatrix}\mid l,m\in N\},$$

then

$$\mathcal{L}_0=\phi_{\varepsilon_1}(\mathcal{L}_0)\cup\phi_{\bar{\varepsilon}_1}(\mathcal{L}_0) \quad ,$$

where $\bar{\varepsilon}_k=1-\varepsilon_k$. Therefore we have a following cone deomposition of $\mathcal{L}_0$ for each x;

$$\mathcal{L}_0=\sum_{k=1}^{\infty}\phi_{\varepsilon_1}\phi_{\varepsilon_2}\cdots\phi_{\varepsilon_{k-1}}\phi_{\bar{\varepsilon}_k}\mathcal{L}_0 \quad .$$

<u>Formula of Legendre constant</u>:

$$\min_{\substack{q>0\\ (q,p):(q,p)=1\\ p/q \text{ is not principle}\\ \text{nor midiant.}}} q|q\alpha-p| \geqq \min\left(\min_{\substack{n:\\ x_n<\frac{1}{2}}}\frac{\left(1+\frac{2r_n}{r_n+s_n}\right)\left(2-3x_n\right)}{\frac{r_n}{r_n+s_n}\left(1-x_n\right)+x_n}, \quad \min_{\substack{n:\\ x_n>\frac{1}{2}}}\frac{\left(2+\frac{r_n}{r_n+s_n}\right)\left(3x_n-1\right)}{\frac{r_n}{r_n+s_n}\left(1-x_n\right)+x_n}\right)$$

<u>Natural extension</u>: put $\bar{X}:=[0,1]\times[0,1]$

and define a map $\bar{T}$ on $\bar{X}$ by

$$\bar{T}(x,y):=\begin{cases}\left(\frac{x}{1-x},\frac{y}{1+y}\right) & \text{if } x\in[0,\frac{1}{2})\\ \left(\frac{1-x}{x},\frac{1}{1+y}\right) & \text{if } x\in[\frac{1}{2},1]\end{cases} \quad ,$$

then $(\bar{X},\bar{T})$ is a natural extension of (X,T) and the fllowing formula holds,

$$\bar{T}^n(x,1)=\left(T_n x,\ \frac{r_n}{r_n+s_n}\right).$$

<u>Result</u>. The Legendre constant c_M associated with mediant convergent transformation is calculated as

$$\min\left(\min_{(x,y)\in[0,\frac{1}{2})\times[0,1]} f(x,y) \quad , \quad \min_{(x,y)\in[\frac{1}{2},1)\times[0,1]} g(x,y)\right)$$

where $f(x,y)=\frac{(1+2y)(2-3x)}{y(1-x)+x}$ and $g(x,y)=\frac{(2+y)(3x-1)}{y(1-x)+x}$

and its value is equal to 1.

So, we obtain the well known result: (q,p) is mediant or principle convergent if $q|q\alpha-p|<1$, $(q,p)=1, q>0$.

Example 4. Even partial quotients transformation

Algorithm: put $X=[0,1)$

$a(x)$ be the nearest even integer of $\frac{1}{x}$

$\varepsilon(x)=\mathrm{sign}\left(\frac{1}{x}-a(x)\right)$

$Tx=\left|\frac{1}{x}-a(x)\right|$

$a_n(x)=a\left(T^{n-1}x\right)$

and $\varepsilon_n(x)=\varepsilon\left(T^{n-1}x\right)$,

then the algrithm (X,T,a,ε) induces an approximation of x by $\frac{v_n}{w_n}$ such that $(w_n,v_n)=(\mathrm{odd},\mathrm{odd})$, where (w_n,v_n) is defined by

$$\begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix}=\begin{pmatrix} a_1 & \varepsilon_1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & \varepsilon_2 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix} .$$

and $w_n=r_n\pm s_n$, $v_n=t_n\pm u_n$.

Cone decomposition : put

$$\phi_n : \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}:=\begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\phi_{n+1,k}: \begin{pmatrix} x_n \\ y_n \end{pmatrix}:=\begin{pmatrix} 2k & \varepsilon_{n+1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$

and

$$\mathscr{L}_0=\left\{(2p+1)\binom{1}{1}+2q\binom{1}{-1},2p\binom{1}{1}+(2q+1)\binom{1}{-1}:q,p\in N\right\}$$
$$\left(=\{(m,n)\mid m,n \text{ odd}, m\geqq 0 \text{ and } |n|\leqq m\}\right)$$

then $\mathscr{L}_0$ has a cone decomposition for each $x\in X$

$$\mathscr{L}_0=\sum_{n=0}^{\infty}\sum_{k\neq a_{n+1}}^{\infty}\mathscr{L}_E(n,k)$$

where

$$\mathscr{L}_E(n,k)=\phi_1\ \phi_2\cdots\phi_n\ \phi_{n+1,k}(\mathscr{L}_0)$$

Formula of Legendre constant:

$$\min_{\substack{(q,p):\\ (q,p)=1\\ (q,p)=(\mathrm{odd},\mathrm{odd})\\ (q,p)\neq(r_n\pm s_n,\, t_n\pm u_n)}} q|qx-p| \geq \min_{0<T^n x<\frac{1}{5}} \frac{\left(3+\frac{s_n}{r_n}\right)\left(1-3\cdot T^n x\right)}{\left(1+\frac{s_n}{r_n}\cdot T^n x\right)}$$

Natural extension : put $\bar{X}:=[0,1)\times[-1,1)$ and define a map on $\bar{X}$ by

$$\bar{T}(x,y)=\left(Tx,\frac{\varepsilon_1}{a_1+y}\right),$$

Then $(\bar{X},\bar{T})$ is a natural extension of (X,T) and the following formula holds

$$\bar{T}^n=(x,0)=\left(T^n x,\frac{s_n}{r_n}\right)$$

Result: Using the above formula and the natural extension,

The Legendre constant c_E associated with even partial quotient is calculated as

$$\min_{\substack{(x,y)\in\bar{X}\\ \text{and } 0<x<\frac{1}{5}}} \frac{(3+y)(1-3x)}{1+y\cdot x}$$

and its value is equal to 1, that is, we have the following result:

if $q|qx-p|<1$, $(q,p)=1$, q,p:odd then there exists n such that

$$(q,p)=\left(r_n\pm s_n,\, t_n\pm s_n\right)$$

References

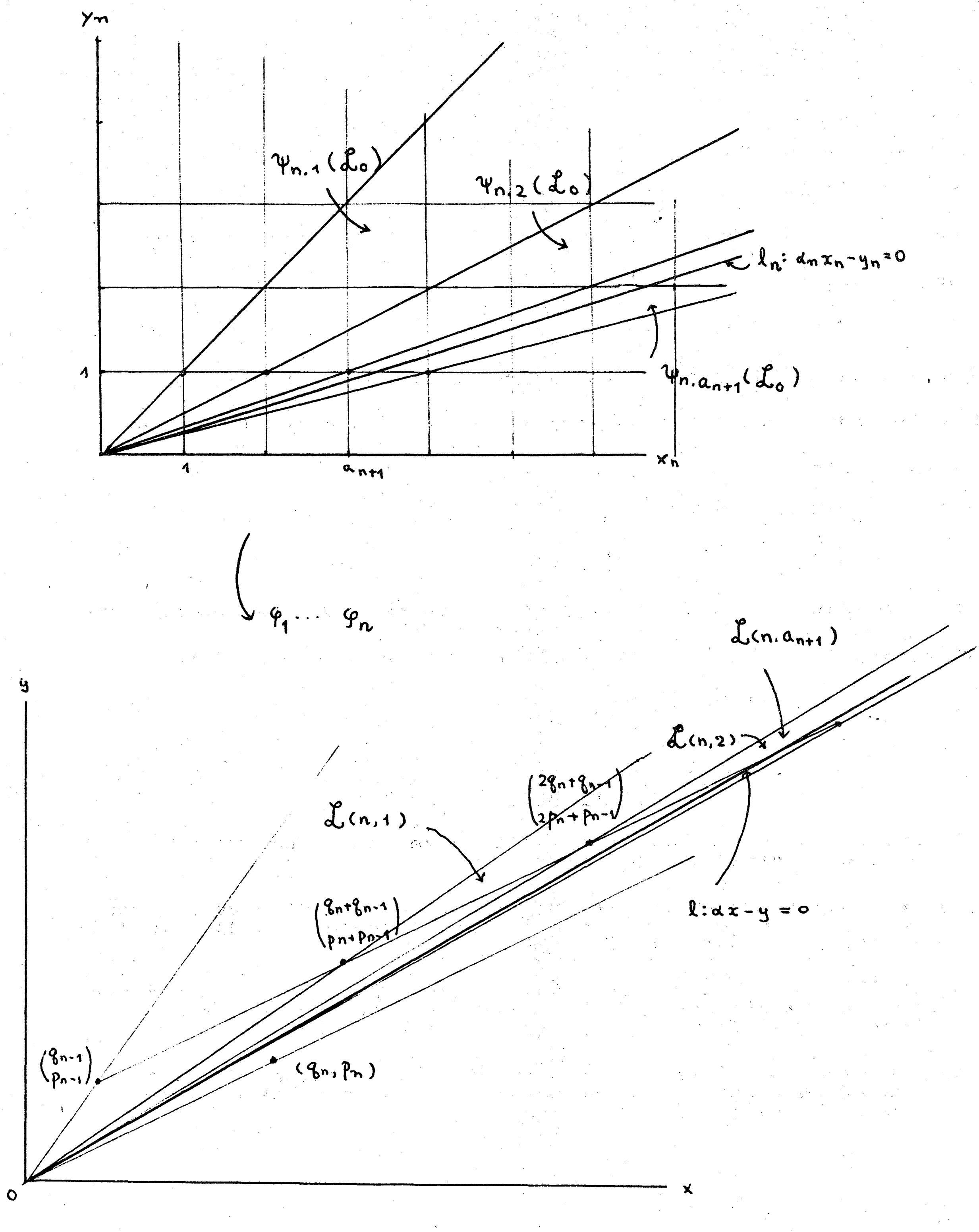
[1] Sh.ITO, Algorithms with mediant convergents and their metrical theory,(preprint).

[2] Sh.ITO, On a Diophantine approximation of real numbers from below,(to appear in Proceedings of Prospects of Mathematical Science, World Scientific Pub. 1988, 43-66.)

[3] Sh.ITO and H.NAKADA, On natural extensions of transformations related to diophantine approximations, Number theory and Conbinatrics, World Scientific Pub.(1985),185-207.

[4] H.NAKADA, Metrical theory for a class of continued fraction transformations and their natural extensions,Tokyo J.Math.4 (1981),399-426.

Figure (1)



y_n
$\psi_{n,1}(\mathcal{L}_0)$
$\psi_{n,2}(\mathcal{L}_0)$
$l_n: \alpha_n x_n - y_n = 0$
$\psi_{n,a_{n+1}}(\mathcal{L}_0)$
1
1
a_{n+1}
x_n
$\varphi_1 \cdots \varphi_n$
$\mathcal{L}(n, a_{n+1})$
y
$\mathcal{L}(n,2)$
$\binom{2q_n+q_{n-1}}{2p_n+p_{n-1}}$
$\mathcal{L}(n,1)$
$l: \alpha x - y = 0$
$\binom{q_n+q_{n-1}}{p_n+p_{n-1}}$
$\binom{q_{n-1}}{p_{n-1}}$
(q_n, p_n)
O
x